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Comments on an ancient Greek racecourse: finding minimum width annuluses

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Abstract For a set of measured points, we describe a linear-programming model that enables us to find concentric circumscribed and inscribed circles whose annulus encompasses all the points and whose width tends to be minimum in a Chebychev minmax sense. We illustrate the process using the data of Rorres and Romano (SIAM Rev. **39**: 745–754, 1997) that is taken from an ancient Greek stadium in Corinth. The stadium's racecourse had an unusual circular arc starting line, and measurements along this arc form the basic data sets of Rorres and Romano (SIAM Rev. **39**: 745–754, 1997). Here we are interested in finding the center and radius of the circle that defined the starting line arc. We contrast our results with those found in Rorres and Romano (SIAM Rev. **39**: 745–754, 1997).

Keywords Circles · Quality control · Chebychev · Minmax · Archaeology · Stadium · Least squares

1 Introduction

The paper by Rorres and Romano [1] calls into play my interest in ancient history, my past running and current jogging activities, and my recent research in quality control that deals with determining whether or not a drilled hole meets its circular "out-of-roundness" specifications.

As posed in [1], the problem is to determine the radius and center of a circular starting line for an ancient (circa 500 B.C.) racetrack found in the Greek city of Corinth. The details of the form and structure of the racetrack are described in Romano [2]. The rectangular racing area was about 600 Corinthian feet in length and about 60 Corinthian feet in width. Excavations in 1980 exposed a 12 m section of a unique curved starting line that formed the northwest end of the racing area. The starting line is made of rectangular blocks of limestone, Fig. 1.

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Fig. 1 Curved starting line of a racecourse in Corinth [1]

The runners took their starting (standing) positions as dictated by 17 sets of front and back toe grooves cut into the limestone. The positions are separated by about one meter. As noted by Romano, the curved starting line was probably used so as to provide a fair start and to equalize the distance for each runner. One can also surmise that judges at the ends of the curve starting line could get a better view of the runners to determine if any "jumped-the-gun." Fig. 1 is a picture of the starting line [1]; also see http://www.cs.drexel.edu/~crorres/Corinth/Stadium.html.

After leaving their starting positions, the runners would move and stay in the right half of the rectangular track until they reached a "turning post" at the further end, around which they would make a left-hand turn that brought them to the right-hand side of the track and back towards a turning post positioned near the starting line. The runners would continue in this manner for the prescribed number of lengths of the race. From a tactical point-of-view, a runner would tend to minimize his distance by keeping to the center of the track and running towards the turning post at each end. (Ancient Greek runners were naked except, possibly, for light weight foot coverings; women were usually banned from these contests. Based on ancient Greek statues and pottery, Romano concludes that Greek runners started in a standing position with their left foot forward. We note that the modern crouch start was not introduced until 1888 [3].)

The problem addressed by Rorres and Romano was to determine the radius and center used by the Corinthian architect in positioning the curved starting line. They give two sets of data points: (1) a series of 21 data points taken along the inside edge of the exposed limestone blocks that form the starting section, and (2) a series of 11 points taken at the center points of the front toe starting position grooves.

In what follows, we first discuss a procedure we applied to measurements of drilled circles taken by coordinate measuring machines and related quality control considerations [4]. We then apply the procedure to the data from [1] and discuss the ramifications and implications of our results on the basic problem of determining the center and radius of the circular starting line.

2 When is a circle a circle? [4]

Mechanical parts are manufactured to meet stated specifications. A surface must be level, a drilled hole circular, and a ball-bearing spherical. But, exactly how level, how circular and how spherical are the manufactured parts? In practice, we need to answer the question: does an object meet the tolerances required by its specifications? The answer to this question is often obtained by measuring the part using a coordinate measuring machine (CMM). A small number of measurements are made which are then analyzed to determine an associated fitted reference object (circle or sphere). If the reference object meets the stated tolerance, the part is said to meet specifications. The tolerance is given in terms of a band width. For example, a drilled hole might have a specified tolerance of 0.4 mm. CMM measurements of the hole, as analyzed by a tolerance procedure (as described below), yield a computed reference circle with radius r_0 and center (x_0, y_0). The hole is within specifications if all the measured points (x_i, y_i) fall within the annulus formed by two concentric circles, the inscribed and circumscribed circles with center at (x_0, y_0), such that the difference between the radii of the inscribed and circumscribed circles is less than or equal to the stated tolerance. A similar discussion of tolerance holds for a sphere in which an "annular shell" is determined.

For a manufactured part, a given set of CMM measurements has an associated "out-of-roundness" value (OOR). The OOR value is determined by selecting a center point for the reference circle, with the value being the difference between the largest and smallest radii to the measured points. The center is calculated by one of the following four alternative methods for determining the circularity of an object:

2.1 Minimum-radial-separation (MRS)

This center is that for which the radial difference between two concentric circles that just contains the measured points is minimum (often referred to as the minimum zone). That is, it is the center which defines an annulus of minimum width, where the annulus is defined by two concentric circles, a circumscribed circle and an inscribed circle. Note that there is an associated reference concentric circle with radius between that of the inscribed and circumscribed circles that, when found, can be used to generate the radii of the inscribed and circumscribed circles and the width of the annulus. The finding of this latter reference circle that generates the minimum annulus is how we shall address the racetrack problem.

The mathematical problem of finding the minimum annulus can be stated as a Chebychev problem in which one wants to minimize the maximum deviation of all the measured points from the circumference of the reference circle. This is a rather complicated nonlinear problem and exact algorithms for solving it are not readily available. In what follows, we show how a reference circle can be determined using standard linear-programming procedures that find an annulus whose width is very close to the width of the true Chebychev annulus [4]. As applied to the two data sets of Rorres and Romano [1], the resulting reference circles would be candidate solutions for their problem

2.2 Least-squares-center (LSC)

This center is that of a circle from which the sum of the squares of the radial ordinates between this circle and the measured points is a minimum. An annulus, similar to the one found by MRS, is formed by the minimum inscribed circle and the maximum circumscribed circle having the least-squares center. Just about all CMMs compute the OOR value from the LSC.

2.3 Maximum-inscribed-circle (MIC/MIS)

This center is that of the largest circle which can be inscribed within the set of measured points. The annulus of interest is formed by this inscribed circle and the concentric circumscribed circle.

2.4 Minimum-circumscribed-circle (MCC/MCS)

This center is that of the smallest circle which just contains all the measured points. The annulus of interest is formed by this circumscribed circle and the concentric inscribed circle.

Algorithms for calculating the above centers exist, with varying computational complexity, Chou et al. [5], Etesami and Qiao [6]. A Voronoi diagram approach can be used to find the MRS center; for LSC, a nonlinear program can be solved; for MIC, a Voronoi diagram method is available; and for MCC, a linear-programming model or a Voronoi diagram algorithm can be used, Hopp and Reeve [7]. Least-squares seems to be the method of choice, although this requires the solution of a set of nonlinear simultaneous equations. However, as stated in ASME [8], the preferred center from which the OOR value is to be determined is the MRS center. We next describe a method for approximating the MRS center as given in [4]. There we show for a large range of CMM test data sets, with 4, 6, 8, 16, 32 and 512 points in each set, that our approximate procedure yields center points and radii and corresponding annuluses all of which are smaller than the least-square annuluses.

3 Fitting a circle by an approximate Chebychev procedure

Chebychev MinMax circle problem
For a set of *n* observations (x_i, y_i), find a center of a circle (x_0, y_0) and radius r_0 such that the maximum radial deviation of the measured points from the determined circumference is minimized.

The approximate minimum-radial separation (AMRS) model and optimality criterion The basic AMRS model is derived from the usual algebraic expression of a circle, that is, $r_0^2 = (x_i - x_0)^2 + (y_i - y_0)^2$. For the set of *n* observations (x_i, y_i), assume that we have calculated a "best-fit" circle with center (x_0, y_0) and radius r_0. We would not expect the (x_i, y_i) to be on the circumference of this circle. That is, the distances r_i of the (x_i, y_i) from the center (x_0, y_0), as measured by $r_i^2 = (x_i - x_0)^2 + (y_i - y_0)^2$, would rarely be equal to r_0. However, we would like the (x_0, y_0, r_0) to be determined in a manner that minimizes the maximum of the absolute differences $|r_i - r_0|$. This form of the problem requires the application of rather complex nonlinear programming or geometric-based algorithms.

In [4], we investigated the effectiveness of the surrogate measure of minimizing the maximum of the absolute differences of the squared terms $|r_i^2 - r_0^2|$. We showed that a solution to the *squared difference problem* by a linear-programming model yields an excellent approximate solution to the unsquared problem for our CMM measured data.

The squared difference problem, minimizing the maximum absolute deviation between the set of r_i^2 and r_0^2, is as follows:

$$\text{Minimize}(x_0, y_0, r_0)\{\text{Maximum}_i |(x_i - x_0)^2 + (y_i - y_0)^2 - r_0^2|\}$$

where (x_0, y_0) is a variable point and the (x_i, y_i) are the measured points, $i = 1, \ldots, n$.

As shown in [4], this problem can be transformed into the linear-programming problem:

$$\text{(P1)} \quad \text{Minimize } t$$

$$\text{subject to } t + 2x_i x_0 + 2y_i y_0 + \rho_0 \geq x_i^2 + y_i^2 \qquad (1)$$

$$t - 2x_i x_0 - 2y_i y_0 - \rho_0 \geq -x_i^2 - y_i^2 \qquad (2)$$

for all i, $t \geq 0$ and x_0, y_0, and $\rho_0 = (r_0^2 - x_0^2 - y_0^2)$ are unrestricted variables.

Solving P1 finds a ρ_0 and center (x_0, y_0) that can be used to determine an OOR value (the width of the annulus between the inscribed and circumscribed circles with center at (x_0, y_0)) for the measured points. From $\rho_0 = (r_0^2 - x_0^2 - y_0^2)$, we determine the associated radius r_0.

The linear-programming problem P1 will have $2n$ inequality constraints and four variables, whereas its dual problem will have four constraints and $2n$ variables. From a computational point of view, as our study involved problems with large n, and the solution time for solving a linear-programming problem is a function of the number of constraints, the dual is of interest. In practice, however, as the number of measured points n is usually small, either the primal or dual formulation could be used effectively. We used the dual as we found it more convenient for data entry and editing, and the version of the linear-programming software that we first used allowed for a larger number of variables than constraints.

Let u_i be the dual variable associated with the ith inequality of constraint set (1), and v_i be the dual variable associated the ith inequality of constraint set (2). The dual problem D1 is then

$$\text{(D1)} \quad \text{Maximize} \quad w = \sum_i (x_i^2 + y_i^2) u_i - \sum_i (x_i^2 + y_i^2) v_i$$

$$\text{subject to} \quad \sum_i u_i + \sum_i v_i \leq 1 \tag{3}$$

$$\sum_i 2x_i u_i - \sum_i 2x_i v_i = 0 \tag{4}$$

$$\sum_i 2y_i u_i - \sum_i 2y_i v_i = 0 \tag{5}$$

$$\sum_i u_i - \sum_i v_i = 0 \tag{6}$$

with $u_i \geq 0$ and $v_i \geq 0$. It can be shown that if an optimal solution to D1 has constraint (3) holding as an inequality, then there is an equivalent solution for which constraint (3) holds as an equation. Thus, in what follows, we consider constraint (3) to be an equation. The solution to P1 or D1 gives us the information to determine the minmax deviation circle. For D1, the dual variable for Eq. 3 is t, the dual variable for Eq. 4 is x_0, the dual variable for Eq. 5 is y_0, and the dual variable for Eq. 6 is ρ_0.

4 The circles of Corinth

We next discuss the application of D1 to the problem and data of Rorres and Romano [1]. Tables 1 and 2 contain the coordinates of the front edge and toe groove positions, respectively. They are based on a reference coordinate system with distances measured in meters.

The positions in Table 1 were located along the front edge of the curved limestone blocks and covered the length of the excavated starting line. The toe groove positions in Table 2 were set in from the front edge along a curve that more or less paralleled the front edge. Toe groove position 5 was indistinct and could not be measured accurately.

Using the data in Table 1 and a least-squares procedure, Rorres and Romano [1] found the following center (x_0, y_0) and radius r_0. We also give the width of the annulus formed by difference between the radii of the concentric circles that circumscribe and inscribe the measured points from the calculated center:

Table 1 Coordinates of 21 data points along the front edge of the starting line [1]

Point #	X	Y
1	19.880	68.874
2	20.159	68.564
3	20.676	67.954
4	20.919	67.676
5	21.171	67.379
6	21.498	66.978
7	21.735	66.692
8	22.810	65.226
9	23.125	64.758
10	23.375	64.385
11	23.744	63.860
12	24.076	63.659
13	24.361	62.908
14	24.597	62.562
15	24.888	62.074
16	25.375	61.292
17	25.166	61.639
18	25.601	60.923
19	25.979	60.277
20	26.180	59.926
21	26.412	59.524

Table 2 Coordinates of the center points of the front toe grooves of 11 of 12 starting positions [1]

Point #	X	Y
1	20.500	69.034
2	20.617	68.426
3	21.168	67.825
4	21.706	67.149
6	22.756	65.723
7	23.354	64.885
8	23.975	63.990
9	24.573	63.042
10	25.122	62.215
11	25.612	61.407
12	26.214	60.421

Table 1 Circles

Least-squares circle [1]:

$x_0 = -20.940,\ y_0 = 33.618\ ;\ r_0 = 53.960\,\mathrm{m}$

Annulus $= 0.053\,\mathrm{m}$

Rorres and Romano also computed centers for 680 combinations of three points from the set of 21 and averaged the results to obtain the following circle:

Three-point average circle [1]:

$x_0 = -22.943,\ y_0 = 32.506;\ r_0 = 56.242\,\mathrm{m}$

Annulus $= 0.084017\,\mathrm{m}$

Using the linear-programming formulation described above, we obtained the following results for the data of Table 1:

Linear-programming minmax circle:

$x_0 = -19.026,\ y_0 = 34.986;\ r_0 = 51.618\,\mathrm{m}$

Annulus $= 0.045\,\mathrm{m}$

Also, using the software package ORDPACK [9], we calculated and alternative least-squares solution, as follows:
ORDPACK least-squares circle:
$x_0 = -20.114$, $y_0 = 34.193$; $r_0 = 52.956$ m
Annulus = 0.0494 m
Using the data in Table 2 (toe-groove starting line), we determined two related circles, as follows (Rorres and Romano did not determine a reference circle for the data of Table 2):

Table 2 Circles
Linear-programming minmax circle:
$x_0 = -16.632059$, $y_0 = 36.641895$; $r_0 = 48.970$ m
Annulus = 0.0654 m
ORDPACK least-squares circle:
$x_0 = -16.329$, $y_0 = 36.831$; $r_0 = 48.613$ m
Annulus = 0.0685 m
As a check on the data, we also calculated circles based on the intersection of perpendicular bisectors of two chords. For the data of Table 1, we used the two chords connecting points 1 and 8 and points 1 and 21. For the data of Table 2, we used the two chords connecting 1 and 12 and 6 and 12.
Perpendicular bisector circle for front-edge data (Table 1):
$x_0 = -17.952$, $y_0 = 35.488$; $r_0 = 50.457$ m
Annulus = 0.0661 m
Perpendicular bisector circle for toe-groove data (Table 2):
$x_0 = -16.867$, $y_0 = 36.102$; $r_0 = 49.471$ m
Annulus = 0.1008 m
In Table 3, we summarize the dimensions and characteristics of the various circles computed for the data in Tables 1 and 2. We also give the radii in Corinthian feet, based on two estimates of the Corinthian foot: one meter equals 3.64 Corinthian feet (Cf1) and one meter equals 3.80 Corinthian feet (Cf2) [2]. The first estimate is equivalent to equating the 600 Corinthian foot racecourse to 165 m, and the second estimate equates the 600 Corinthian feet to 158 m. The 165 and 158 lengths are bounds for the length of the Corinth racecourse, as described in [2].

Based on archeological information [2], the length of the racecourse in Corinth is assumed to be a stadion, that is, 600 Corinthian feet. Depending on the exact length of the Corinthian foot, the length of the racecourse in Corinth should be between 158 and 165 m [2]. In [2], Romano describes how the Corinth racecourse was laid out. The starting line arc was traced out by the end of a radius 200 Corinthian feet long (one-third the length of the racecourse), with the circle's center positioned off the center line of the racecourse. Assuming that the stadion is equal to 165 m (3.64 Corinthian feet per meter), then, from the data in Table 1, the radius of the least-squares solution computed in [1] comes close to being one-third of the length of the racecourse, that is 0.327 (196.41 Corinthian feet). However, if we use the lower estimate of the length of the racecourse, 158 m (3.80 Corinthian feet per meter), then the LP minmax radius comes close to being one-third of the length of the racecourse, also 0.327 (196.15 Corinthian feet). The ORDPACK least-squares solution comes closest to the required 0.333 for the 158 m length (0.335 or 201.24 Corinthian feet). We summarize the distances of the various radii in terms of the fraction of the total length of the racecourse for the 165 and 158 m lengths in Table 4.

Also, from Table 3, we note the following. One would think that the Greek architect/surveyor who laid out the track would be concerned more with having the front toe grooves

Table 3 Summary of circles

Circle	x_0	y_0	r_0 (m)	Radius in Corinthian feet	Annulus (m)
Table 1 Data					
Least Squares [1]	−20.940	33.618	53.960	196.41 Cf1 205.05 Cf2	0.053
3-Point Avg. [1]	−22.943	32.506	56.242	204.72 Cf1 213.72 Cf2	0.084
LP MinMax	−19.026	34.986	51.618	187.98 Cf1 196.15 Cf2	0.045
Least Squares ORDPACK	−20.114	34.193	52.956	192.76 Cf1 201.24 Cf2	0.049
Bisectors	−17.952	35.488	50.457	183.66 Cf1 191.74 Cf2	0.066
Table 2 Data					
LP MinMax	−16.632	36.642	48.970	178.25 Cf1 186.09 Cf2	0.065
ORDPACK Least Squares	−16.329	36.831	48.613	176.95 Cf1 184.73 Cf2	0.069
Bisectors	−16.867	36.102	49.471	180.07 Cf1 187.99 Cf2	0.101

1 m = 3.64 Cf1; 1 m = 3.80 Cf2

Table 4 Radii of circles in terms of fraction of racecourse length

Circle	r_0	r_0/racecourse length		Annulus (m)
		Length = 165 m	Length = 158 m	
Table 1 Data				
Least Squares [1]	53.960 m 196.41 Cf 1 205.05 Cf2	0.327	0.342	0.053
3-Point Avg. [1]	56.242 m 204.72 Cf1 213.72 Cf2	0.341	0.356	0.084
LP MinMax	51.618 m 187.89 Cf1 196.15 Cf2	0.313	0.327	0.045
ORDPACK Least Squares	52.956 m 192.76 Cf1 201.24 Cf2	0.321	0.335	0.049
Bisectors	50.457 m 183.66 Cf1 191.74 Cf2	0.306	0.319	0.066
Table 2 Data				
LP MinMax	48.970 m 178.25 Cf1 186.09 Cf2	0.297	0.310	0.065
ORDPACK Least Squares	48.613 m 176.95 Cf 1 184.73 Cf2	0.295	0.308	0.069
Bisectors	49.471 m 180.07 Cf1 187.99 Cf2	0.300	0.313	0.101

1 m = 3.64 Cf1; 1 m = 3.80 Cf2

lined-up on a true circular arc than with the edge of the limestone blocks that defined the starting line. We can imagine that the limestone blocks were first laid at the beginning of the race course and then the toe grooves were cut on a circular arc swept out by the end of a radius of appropriate length. Given the toe grooves, then the front and back edges edge of the limestone blocks would be shaped to conform to the toe-groove arc and any excess limestone removed. Note that the Table 2 circles have centers and radii that are relatively close, in contrast to the circles for the data of Table 1 (This is more so if we ignore the bisector and three-point average circles, as they tend to be less accurate and have the largest annuluses). However, the length of the associated radii are the furthest away from the desired 0.333 ratio for all the Table 2 circles. Rorres and Romano [1] base their calculations on the 21 measurements taken at the edge of the limestone blocks (Table 1); they did not determine the circle for the toe-groove arc (Table 2). It is interesting to note that the radius for the toe-groove arc is 3–4 m shorter than the circle for the limestone block edge arc. This could be due to the irregularity of both the front-edge and toe-groove arcs, as noted by Romano [2].

It is, of course, difficult to conclude which circle yields the best approximation of the architect's true center and radius. We suggest, based on the tighter annuluses and the ratios of the radii to the two extreme lengths, 165 and 158 m, that the LP MinMax is an appropriate choice, given one assumes a station length of 158 m.

References

1. Rorres, C., Romano, D.G.: Finding the center of a circular starting line in an ancient Greek stadium. SIAM Rev. **39**, 745–754 (1997). Also see http://www.cs.drexel.edu/~crorres/Corinth/Stadium.html (viewed on June 11, 2007).
2. Romano, D.G.: Athletes and Mathematics in Archaic Corinth: The Origins of the Greek Station, Vol. 26. American Philosophical Society, Philadelphia, PA (1993)
3. Conger, R.M.: Track and Field. A. S. Barnes and Co., New York (1939)
4. Gass, S.I., Witzgall, C., Harary, H.H.: Fitting circles and spheres to coordinate measuring machine data. Int. J. Flex. Manuf. Syst. **10**, 5–25 (1998)
5. Chou, S.-Y., Woo, T.C., Pollock, S.M.: On characterizing circularity. Department of Industrial and Operations Research, University of Michigan, Ann Arbor, MI (1994)
6. Etesami, F., Qiao, H.: Analysis of two-dimensional measurement data for automated inspection. J. Manuf. Syst. **7**, 223–232 (1988)
7. Hopp, T.H., Reeve, C.A.: An algorithm for computing the minimum covering sphere in any dimension. NISTIR 5831, National Institute of Standards and Technology, Gaithersburg, MD (1996)
8. American Society of Mechanical Engineers: Measurement of out-of-roundness. ANSI B89.3.1-1972, New York, NY (1972)
9. Boggs, P.T., Rogers, J.E.: Orthogonal distance regression. Contemp. Math. **112**, 183–194 (1990)